IMPROVED ENERGY-MOMENTUM CURRENTS IN METRIC-AFFINE SPACETIME

by

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Abstract

In Minkowski spacetime it is well-known that the canonical energy-momentum current is involved in the construction of the globally conserved currents of energy-momentum and total angular momentum. For the construction of conserved currents corresponding to (approximate) scale and proper conformal symmetries, however, an improved energy-momentum current is needed. By extending the Minkowskian framework to a genuine metric-affine spacetime, we find that the affine Noether identities and the conformal Killing equations enforce this improvement in a rather natural way. So far, no gravitational dynamics is involved in our construction. The resulting dilation and proper conformal currents are conserved provided the trace of the energy-momentum current satisfies a (mild) scaling relation or even vanishes.

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1. **Introduction**

If a spacetime admits symmetries, we can construct a set of *invariant* conserved quantities, one for each symmetry. Consider the Riemannian spacetime of general relativity (GR). By a *Killing symmetry* we understand a diffeomorphism of the Riemannian spacetime under which the metric is invariant. Let the vector field $\xi = \xi^\alpha\, e_\alpha$ be the generator of such a 1-parameter group of diffeomorphisms of the spacetime. Then it obeys the Killing equation[(1)]

$$\mathcal{L}_\xi\, g = 0 \qquad \Rightarrow \qquad e_{(\alpha} \rfloor \overset{o}{D} \xi_{\beta)} = 0\,, \tag{1.1}$$

with the Riemannian exterior covariant derivative $\overset{o}{D}$.

Following the standard procedure of GR, see Penrose and Rindler [1], it is straightforward to construct an energy-momentum current which is a closed form, provided the matter field equation is satisfied. Let $t_{\alpha\beta}$ be the symmetric energy-momentum tensor of matter and t_α the corresponding covariantly conserved energy-momentum 3-form[(2)] . Then the energy current $\varepsilon_{\,\mathrm{R}}$ represents a weakly closed 3-form:

$$\varepsilon_{\,\mathrm{R}} := \xi^\alpha\, t_\alpha\,, \qquad\qquad d\,\varepsilon_{\,\mathrm{R}} \cong 0\,. \tag{1.2}$$

Weakly means that the last equation is only valid, if the matter field equation holds. Accordingly, for a timelike Killing vector field, $\varepsilon_{\,\mathrm{R}}$ yields, after integration over a spacelike hypersurface S, the conserved energy $\int_S \varepsilon_{\,\mathrm{R}}$.

For a bosonic description of gravitational interactions, the metric-affine gauge theory of gravity (MAG) is a very general framework, see Ref.[2]. The spacetime continuum of MAG is represented by a differentiable manifold (L_4, g) with a coframe ϑ^α, an arbitrary linear connection $\Gamma_\alpha{}^\beta$, and a second rank symmetric tensor field, the metric $g = g_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta$. In extending the previous notion, we will now understand by a Killing symmetry a diffeomorphism under which the metric *and* the connection are invariant.

The aim of this paper is twofold: First we want to generalize the notion of a weakly conserved energy current à la (1.2) to the metric-affine spacetime of MAG. In Sec.4, we will derive such a current $\varepsilon_{\,\mathrm{MA}}$, which generalizes the $\varepsilon_{\,\mathrm{R}}$ of (1.2).

Moreover, in Sec.5 we will relax the Killing constraints on the vector field ξ and consider *conformal* Killing vector fields in order to obtain also conserved dilation and proper conformal currents. In Minkowski spacetime, this construct is well-known.

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(1) Here e_α is an arbitrary vector basis of the tangent space and $g_{\alpha\beta}$ are the components of the metric. The Lie derivative with respect to ξ is denoted by $\mathcal{L}_\xi$, the covariant exterior derivative by D, and the interior product by $\rfloor$.

(2) If ϑ^α is the 1-form basis dual to e_α, we have $t_\alpha = t_{\alpha\beta}\,\eta^\beta$ where $\eta^\alpha = {}^*\vartheta^\alpha$ and $*$ denotes the Hodge star. Moreover, $\eta^{\alpha\beta} = {}^*(\vartheta^\alpha \wedge \vartheta^\beta)$.

Here we are generalizing it for the first time to MAG, in the framework of which it finds its most natural embedding.

In Minkowski spacetime (cf. Jackiw[3])

$$\mathcal{D}^\alpha = x^\gamma \, \tilde{t}_\gamma{}^\alpha \,, \tag{1.3}$$

$$\mathcal{K}^{\alpha\beta} = [2x^\beta x^\gamma - g^{\beta\gamma} x^2] \, \tilde{t}_\gamma{}^\alpha \tag{1.4}$$

are the dilation current and the proper conformal current, respectively, if $\tilde{t}^{\alpha\beta}$ represents the improved energy-momentum tensor. From the divergence of these currents we see that both, scale and proper conformal invariance, are broken by the trace $\tilde{t}_\mu{}^\mu$ of the improved energy tensor. Since MAG is formulated in a locally dilation invariant way, we expect that an appropriately constructed conformal current $\varepsilon_{\,\mathrm{C}}$ emerges in terms of a conformal Killing vector. This expression is displayed in Eq.(6.10). We will prove that it is 'weakly' conserved, provided the energy-momentum is tracefree. Therefrom, in Sect.6, we will construct an 'improved' energy–momentum current $\tilde{\varepsilon}_{\,\mathrm{MA}}$ which has a 'soft' trace also for scalar fields, as exemplified by the dilaton, provided the mild constraint (6.11) holds. All the previously discussed energy–momentum expressions can be recovered as special cases of our new expressions (5.10) or (6.10). We conclude our paper in Sec.7 with a relevant application of the Ogievetsky theorem.

2. Metric-affine spacetime in brief

The geometrical variables of such a metric-affine spacetime are the forms $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$ with an appropriate transformation behavior under the local $GL(n,R)$ group in n dimensions. The components $g_{\alpha\beta}$ of the metric g are 0-forms, the coframe and the connection are 1-forms. In MAG, the *nonmetricity*

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} \quad \Rightarrow \quad Q^{\alpha\beta} = Dg^{\alpha\beta} \,, \tag{2.1}$$

besides torsion T^α and curvature $R_\alpha{}^\beta$, enters the spacetime arena as a new field strength. The gauge covariant derivative

$$D\Psi = d\Psi + {}^{\dagger}\Gamma_\alpha{}^\beta \wedge \rho(L_\beta{}^\alpha)\Psi + \frac{w}{2} Q \wedge \Psi \tag{2.2}$$

contains the volume–preserving $SL(n,R)$ connection ${}^{\dagger}\Gamma_\alpha{}^\beta$ and a volume–changing piece depending on the Weyl 1-form $Q := (1/n)\, Q^\alpha{}_\alpha$. The matter fields are allowed to be densities, that is, under the scale part $\Lambda_\alpha{}^\beta = \delta^\beta_\alpha \, \Omega$ of the $GL(n,R)$ gauge transformation the matter field will transform as $\Psi' = \Omega^w \, \Psi$. The weight w depends on the $SL(n,R)$ index structure and the density character. Our notation is the same as in Refs. [2].

potential	**field strength**	**Bianchi identity**
metric $g_{\alpha\beta}$	$Q_{\alpha\beta} = -Dg_{\alpha\beta}$	$DQ_{\alpha\beta} = 2R_{(\alpha\beta)}$
coframe ϑ^α	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha = R_\mu{}^\alpha \wedge \vartheta^\mu$
connection $\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta + \Gamma_\mu{}^\beta \wedge \Gamma_\alpha{}^\mu$	$DR_\alpha{}^\beta = 0$

Following Wallner [4], we use $\mathcal{L}_\xi$ to denote the usual Lie derivative of a general geometric object and ℓ_ξ to denote the restriction of the Lie derivative to differential forms, for which one can show that

$$\ell_\xi := \xi \rfloor d \; + \; d\, \xi \rfloor \, . \tag{2.3}$$

The "gauge covariant Lie derivative" of a form is given by

$$\text{Ł} := \xi \rfloor D \; + \; D\, \xi \rfloor \, . \tag{2.4}$$

As an example, we obtain for the connection

$$\mathcal{L}_\xi \Gamma_\alpha{}^\beta = (\ell_\xi - \delta_\xi)\, \Gamma_\alpha{}^\beta \qquad \text{with} \qquad \delta_\xi \Gamma_\alpha{}^\beta = -D\left(e_\alpha \rfloor (\ell_\xi \vartheta^\beta)\right) . \tag{2.5}$$

Later on, we will employ the covariant exterior derivative $\overset{\frown}{D}$ with respect to the *transposed* connection

$$\overset{\frown}{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta \, . \tag{2.6}$$

Its somewhat unclear role becomes more transparent by the following observation: If applied to the vector components ξ^α, the transposed derivative is identical to the gauge-covariant Lie derivative of the coframe, i.e.

$$\text{Ł}_\xi\, \vartheta^\alpha \equiv \overset{\frown}{D}\, \xi^\alpha \, . \tag{2.7}$$

3. Lagrange-Noether machinery

The external currents of a matter field are those currents which are related to *local* spacetime symmetries. On a fundamental level, we adopt the view that fundamental matter is described in terms of infinite-dimensional spinor or tensor representations of $SL(4,R)$, the *manifields* Ψ, see Ne'eman [5].

In a first order formalism (cf. Nester [6] and Kopczyński [7]) we assume that the material Lagrangian n-form for these manifields depends most generally on Ψ, $d\Psi$, and the potentials $g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$. According to the minimal coupling prescription, derivatives of these potentials are not permitted. We usually adhere to this principle. However, Pauli type terms and the Jordan-Brans-Dicke term $\bar{\Phi}\Phi\, R^{\alpha\beta} \wedge \eta_{\alpha\beta}$ may occur in conformal models with 'improved' energy-momentum tensors or in the context of symmetry breaking. Therefore, we have developed [2] our Lagrangian formalism in sufficient generality in order to cope with such models by including in the Lagrangian also the derivatives $dg_{\alpha\beta}$, $d\vartheta^\alpha$, and $d\Gamma_\alpha{}^\beta$ of the gravitational potentials:

$$L = L(g_{\alpha\beta}\,, dg_{\alpha\beta}\,, \vartheta^\alpha\,, d\vartheta^\alpha\,, \Gamma_\alpha{}^\beta\,, d\Gamma_\alpha{}^\beta\,, \Psi, d\Psi)\,. \tag{3.1}$$

As a further bonus, we can then easily read off the Noether identities for the gravitational gauge fields in $n > 2$ dimensions (the restriction is related to the introduction of conformal symmetry in Sec. 5).

For such a gauge-invariant Lagrangian L, the variational derivative $\delta L/\delta\Psi$ becomes identical to the $GL(n,R)$-covariant *variational derivative* of L with respect to the q-form Ψ:

$$\frac{\delta L}{\delta \Psi} := \frac{\partial L}{\partial \Psi} - (-1)^q D \frac{\partial L}{\partial (D\Psi)}\,. \tag{3.2}$$

The *matter currents* are the metric stress-energy $\sigma^{\alpha\beta}$, the canonical energy-momentum Σ_α, and the hypermomentum $\Delta^\alpha{}_\beta$, which is asymmetric in α and β. They are given by

$$\sigma^{\alpha\beta} := 2\frac{\delta L}{\delta g_{\alpha\beta}} = 2\frac{\partial L}{\partial g_{\alpha\beta}} + 2D\frac{\partial L}{\partial Q_{\alpha\beta}}\,, \tag{3.3}$$

$$\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} = \frac{\partial L}{\partial \vartheta^\alpha} + D\frac{\partial L}{\partial T^\alpha}\,, \tag{3.4}$$

and

$$\begin{aligned}\Delta^\alpha{}_\beta := \frac{\delta L}{\delta \Gamma_\alpha{}^\beta} &= L^\alpha{}_\beta\,\Psi \wedge \frac{\partial L}{\partial (D\Psi)} \\ &\quad + 2g_{\beta\gamma}\,\frac{\partial L}{\partial Q_{\alpha\gamma}} + \vartheta^\alpha \wedge \frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial R_\alpha{}^\beta}\,.\end{aligned} \tag{3.5}$$

The explicit form of the *canonical energy-momentum current* reads

$$\begin{aligned}\Sigma_\alpha = e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi} - (e_\alpha\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} \\ - (e_\alpha\rfloor Q_{\beta\gamma})\,\frac{\partial L}{\partial Q_{\beta\gamma}} - (e_\alpha\rfloor T^\beta)\wedge\frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\frac{\partial L}{\partial R_\beta{}^\gamma}\,.\end{aligned} \tag{3.6}$$

The first line in (3.6) represents the result known in the context of *special* relativistic classical field theory. The last Ψ-dependent term in (3.6) vanishes for a 0-form, as is exemplified by a scalar or the Dirac field. The second line in (3.6) accounts for possible Pauli terms and is absent in the case of minimal coupling.

In the following, our Noether currents, aside from corresponding to the gauge action of $GL(n,R)$, involve local translations as well. In Minkowski spacetime, we could restrict the transformations to constant parameters and apply $A(n,R) = R^n \otimes GL(n,R)$ globally, still obtaining the familiar conservation laws, including those of the dilations and conformal transformations (1.3, 1.4). In our present paper, however, we treat the whole $A(n,R)$ locally, even though one should observe that the "gauging" of R^n transcends the definition of a Lie algebra and the corresponding bundle Maurer-Cartan equations.

The *first Noether identity* in MAG takes the form

$$\begin{aligned} D\Sigma_\alpha \equiv & (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\, \sigma^{\beta\gamma} \\ & + (e_\alpha \rfloor D\Psi)\, \frac{\delta L}{\delta \Psi} + (-1)^p (e_\alpha \rfloor \Psi) \wedge D\frac{\delta L}{\delta \Psi} \\ \cong & (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\, \sigma^{\beta\gamma}. \qquad \text{(1st)} \end{aligned} \tag{3.7}$$

Usually, our first result is given in the *strong* form, where no field equation is invoked. The *weak* identity, which is denoted by $\cong$, holds only provided the matter field equation $\delta L/\delta\Psi = 0$ is satisfied.

The *second Noether identity* in MAG reads

$$D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - g_{\beta\gamma}\, \sigma^{\alpha\gamma} \equiv -L^\alpha{}_\beta \Psi \wedge \frac{\delta L}{\delta \Psi} \cong 0\,. \qquad \text{(2nd)} \tag{3.8}$$

The dilational part of the *second* Noether identity can be easily extracted directly from (3.8) by sheer contraction:

$$D\Delta + \vartheta^\alpha \wedge \Sigma_\alpha - \sigma^\alpha{}_\alpha \equiv -L^\gamma{}_\gamma \Psi \wedge \frac{\delta L}{\delta \Psi} \cong 0\,. \tag{3.9}$$

Only the trace piece $\vartheta^\alpha \wedge \Sigma_\alpha$ of the energy-momentum current contributes to this dilational identity, cf. [8, 9]. This identity plays an important role for the approximate scale invariance in the high-energy-limit of particle physics (see [10]) and in the construction of the improved current.

4. **Conserved currents in MAG**

In MAG, the connection is an independent field variable. The corresponding matter current, coupled to it, will be the hypermomentum $\Delta^\alpha{}_\beta$. We require $\xi = \xi^\alpha\, e_\alpha$ to be a Killing vector field for metric *and* connection,

$$\mathcal{L}_\xi\, g = \left(\text{Ł} g_{\alpha\beta} + 2 g_{\gamma(\alpha} e_{\beta)} \rfloor \text{Ł}\vartheta^\gamma\right)\; \vartheta^\alpha \otimes \vartheta^\beta = 0\,, \qquad \mathcal{L}_\xi \Gamma_\alpha{}^\beta = 0\,. \tag{4.1}$$

According to (2.3), (2.5), and (2.7), these conditions can be recast into the form

$$g_{\gamma(\alpha}\, e_{\beta)} \rfloor \widehat{D}\, \xi^\gamma - \frac{1}{2}\xi \rfloor Q_{\alpha\beta} = 0\,, \qquad D(e_\alpha \rfloor \widehat{D}\, \xi^\beta) + \xi \rfloor R_\alpha{}^\beta = 0\,. \tag{4.2}$$

Note that the first equation of (4.2) can be written alternatively, in terms of the Riemannian derivative, as $e_{(\alpha}\rfloor\overset{o}{D}\xi_{\beta)} = 0$, compare the second equation of (1.1).

Let us define the current $(n-1)$-form

$$\varepsilon_{\rm MA} := \xi^\alpha\,\Sigma_\alpha + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,\Delta^\beta{}_\gamma \quad . \tag{4.3}$$

We compute its exterior covariant derivative, substitute the two Noether identities (3.7) and (3.8), and reshuffle the emerging expressions:

$$\begin{aligned} d\,\varepsilon_{\rm MA} &= (D\xi^\alpha)\wedge\Sigma_\alpha + \xi^\alpha\,D\Sigma_\alpha + D(e_\beta\rfloor\widehat{D}\xi^\gamma)\wedge\Delta^\beta{}_\gamma + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,D\Delta^\beta{}_\gamma \\ &\cong (D\xi^\alpha)\wedge\Sigma_\alpha + (\xi\rfloor T^\beta)\wedge\Sigma_\beta + (\xi\rfloor R_\beta{}^\gamma)\wedge\Delta^\beta{}_\gamma - \frac{1}{2}(\xi\rfloor Q_{\beta\gamma})\,\sigma^{\beta\gamma} \\ &\quad + D(e_\beta\rfloor\widehat{D}\xi^\gamma)\;\wedge\Delta^\beta{}_\gamma + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,(\sigma^\beta{}_\gamma - \vartheta^\beta\wedge\Sigma_\gamma) \\ &= \left[\widehat{D}\xi^\alpha - \vartheta^\beta\,(e_\beta\rfloor\widehat{D}\xi^\alpha)\right]\wedge\Sigma_\alpha + \left[(De_\beta\rfloor\widehat{D}\xi^\gamma) + \xi\rfloor R_\beta{}^\gamma\right]\wedge\Delta^\beta{}_\gamma \\ &\quad + \left[g_{\alpha(\beta}\,e_{\gamma)}\rfloor\widehat{D}\,\xi^\alpha - \frac{1}{2}\xi\rfloor Q_{\beta\gamma}\right]\sigma^{\beta\gamma}\,. \end{aligned} \tag{4.4}$$

While transforming the exterior derivative into the gauge covariant derivative (2.2), we assumed that ε has zero weight w_ε, which is in accordance with the weight zero for the Lagrangian and a usual vector field ξ. Moreover, we recognize that the first square bracket vanishes identically because of the relation $p\psi = \vartheta^\alpha\wedge(e_\alpha\rfloor\psi)$, which is valid for any p-form. In view of the generalized Killing equations (4.2), also the other expressions in the square brackets vanish. Thus, the current (4.3) is weakly conserved

$$d\,\varepsilon_{\rm MA} \cong 0\,. \tag{4.5}$$

For the Riemann-Cartan spacetime of the Einstein-Cartan-Sciama-Kibble theory a similar result has been obtained by Trautman [11] and, for the linearized case, by Tod [12]. The corresponding current reads

$$\varepsilon_{\rm RC} := \xi^\alpha\,\Sigma_\alpha + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,\tau^\beta{}_\gamma\,, \tag{4.6}$$

where the spin current is defined according to $\tau^{\beta\gamma} := \Delta^{[\beta\gamma]}$. In Audretsch et al. [13], this current was used to construct a Hamiltonian for the Dirac field.

Provided a timelike Killing vector field exists, we have obtained, via (4.5), a globally conserved energy $\int_S \varepsilon_{\rm MA}$. Our deduction of this expression follows the pattern laid out in GR, but generalizes it to a metric-affine spacetime. Some steps of this deduction resemble Penrose's recent *local mass construction* [14], except that we refrain from using spinor or twistor methods at this stage.

5. **Conserved dilation and proper conformal currents**

If the metric-affine spacetime admits a conformal symmetry, an important generalization of (4.3) can be constructed as follows: Let $\xi = \xi^\alpha \, e_\alpha$ be a conformal Killing vector field such that the Lie derivative of the metric g and the connection $\Gamma_\alpha{}^\beta$ read[(3)]

$$\mathcal{L}_\xi \, g = \omega \, g \,, \qquad\qquad \mathcal{L}_\xi \Gamma_\alpha{}^\beta = \frac{1}{2}\delta^\beta_\alpha \, d\omega \,. \tag{5.1}$$

The same algebra as in Sect.4 yields

$$g_{\gamma(\alpha} \, e_{\beta)} \rfloor \widehat{D} \, \xi^\gamma - \frac{1}{2}\xi\rfloor Q_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \, \omega \,, \qquad\qquad D(e_\alpha \rfloor \widehat{D} \, \xi^\beta) + \xi\rfloor R_\alpha{}^\beta = \frac{1}{2}\delta^\beta_\alpha \, d\omega \,, \tag{5.2}$$

compare with (4.2), which we recover for $\omega = 0$. It follows from $(5.1)_1$ that ξ generates a transformation, parametrized by λ, of the spacetime mainfold such that the metric undergoes the special [15] conformal change $g \to \tilde{g} = e^{\lambda\omega} \, g$. For a given geometry, the scalar function $\omega = \omega(x)$ is determined by the trace of $(5.2)_1$, i.e. by

$$\omega = \frac{2}{n} e_\gamma \rfloor \widehat{D}\xi^\gamma - \xi\rfloor Q \,. \tag{5.3}$$

Thus, in metric-affine spacetime the conformal Killing equation for the metric reads

$$g_{\gamma(\alpha} \, e_{\beta)} \rfloor \widehat{D} \, \xi^\gamma - \frac{1}{n} g_{\alpha\beta} \, e_\gamma \rfloor \widehat{D}\xi^\gamma = \frac{1}{2}\xi\rfloor \not{Q}_{\alpha\beta} \,, \tag{5.4}$$

where $\not{Q}_{\alpha\beta} := Q_{\alpha\beta} - g_{\alpha\beta} \, Q$ is the tracefree part of the nonmetricity.

Again we compute the exterior derivative of (4.3), but now under the assumption that the vector field ξ is a conformal Killing field obeying (5.1) or (5.2), respectively. Then the expressions in the last two square brackets in (4.4) do not vanish any more. Rather we find with the aid of the Noether identity (3.9) the relation

$$d \, \varepsilon_{\,\mathrm{MA}} \cong \frac{1}{2} \, D\omega \wedge \Delta^\alpha{}_\alpha + \frac{1}{2}\omega \, \sigma^\alpha{}_\alpha \;\cong d \, (\frac{1}{2}\omega \, \Delta) + \frac{1}{2}\omega \, (\vartheta^\alpha \wedge \Sigma_\alpha) \,. \tag{5.5}$$

In the case of vanishing ω, we recover (4.5). If ω does not vanish, eq.(5.5) yields

$$d \, (\varepsilon_{\,\mathrm{MA}} - \frac{1}{2}\omega \, \Delta) \cong \frac{1}{2}\omega \, (\vartheta^\alpha \wedge \Sigma_\alpha) \,. \tag{5.6}$$

For conformally invariant gauge theories, such as the Maxwell or the Yang-Mills vacuum theory, the trace $\vartheta^\alpha \wedge \Sigma_\alpha$ of the energy-momentum current vanishes and (5.6) provides already the conserved quantity

$$\varepsilon_{\,\mathrm{C}} := \varepsilon_{\,\mathrm{MA}} - \frac{1}{2}\omega \, \Delta \,. \tag{5.7}$$

[(3)] Eq. $(5.1)_2$ implies the vanishing of the trace free part of $(4.2)_2$. The same would hold, too, by requiring $(5.1)_2$ for the volume–preserving connection instead.

Since the traces of the energy-momentum current and the hypermomentum current $(n-1)$–forms are crucial here, we decompose these currents into their tracefree pieces and their traces, respectively:

$$\Sigma_\alpha = \not{\Sigma}_\alpha + \frac{1}{n}\, e_\alpha\rfloor(\vartheta^\gamma \wedge \Sigma_\gamma)\,, \qquad\qquad \Delta^\alpha{}_\beta = \not{\Delta}^\alpha{}_\beta + \frac{1}{n}\,\delta^\beta_\alpha\,\Delta\,. \tag{5.8}$$

We substitute first (4.3) and then (5.8) into (5.7). This yields

$$\varepsilon_{\,\mathrm{C}} = \xi^\alpha \not{\Sigma}_\alpha + \frac{1}{n}\,\xi\rfloor(\vartheta^\alpha \wedge \Sigma_\alpha) + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,\not{\Delta}^\beta{}_\gamma + \frac{1}{n}\left[(e_\alpha\rfloor\widehat{D}\xi^\alpha) - \frac{n}{2}\omega\right]\,\Delta\,. \tag{5.9}$$

If we apply the trace (5.3) of one of the conformal Killing relations, we obtain

$$\varepsilon_{\,\mathrm{C}} = \xi^\alpha \not{\Sigma}_\alpha + (e_\beta\rfloor\widehat{D}\xi^\gamma)\,\not{\Delta}^\beta{}_\gamma + \frac{1}{n}\,\xi\rfloor(\vartheta^\alpha \wedge \Sigma_\alpha) + \frac{1}{2}(\xi\rfloor Q)\,\Delta\,. \tag{5.10}$$

Thus we have found generalizations of the well-known dilation and proper conformal currents in Minkowski spacetime (cf. Ref. [3]) to a metric-affine spacetime. Such a spacetime provides the most natural gravitational background for these currents.

6. Improved currents with a 'soft' energy–momentum trace for scalar fields

Following Isham et al. [16], we consider the so-called *dilaton field* $\sigma(x)$ being immersed into our MAG framework.

On the classical level, we can assume that the scalar field carries canonical dimensions , i.e. $(length)^{-1}$ in $n = 4$ dimensions. With respect to a generic conformal change $g \to \tilde{g} = \Omega(x) g$ of the underlying metric structure, the scalar field will then transform according to

$$\sigma(x) \to \tilde{\sigma}(x) = \Omega(x)^{-(n-2)/4}\sigma(x)\,, \tag{6.1}$$

i.e. as a scalar density of weight $w_\sigma = (2-n)/2$. Then the gauge–covariant exterior derivative (2.2) is given by

$$D\sigma := \left(d - \frac{n-2}{4}\,Q\right)\sigma \tag{6.2}$$

which, due to the inhomogeneous transformation of the Weyl 1–form under conformal changes, is conformally covariant.

The dilaton Lagrangian with a polynomial self–interaction n–form $V(|\sigma|)$ is given by

$$L_\sigma = L_\Box + V(|\sigma|), \quad L_\Box := \frac{1}{2}\,D\sigma \wedge {}^*D\sigma\;. \tag{6.3}$$

Eq. (6.2) implies that its kinetic part is conformally invariant in any dimensions. By variation of (6.3) with respect to σ, we obtain the scalar wave equation

$$\frac{\delta L_\sigma}{\delta\sigma} = \Box\,\sigma + \frac{\partial V(|\sigma|)}{\partial\sigma} = 0, \tag{6.4}$$

where

$$\Box\sigma := -D\,{}^*(D\sigma) \tag{6.5}$$

is the d'Alembertian.

The trace of the dilaton's energy–momentum current, i.e.

$$\vartheta^\alpha \wedge \Sigma_\alpha(\sigma) = (n-2)\,L_\Box + nV(|\sigma|) \tag{6.6}$$

does not vanish. Therefore, we need to "improve" Σ_α in this respect. Since the kinetic part of the dilaton Lagrangian (6.3) depends explicitly on the Weyl 1–form Q, the scalar field does also provide an intrinsic dilation current. According to (3.5), the latter is dynamically defined by

$$\Delta := \Delta^\gamma{}_\gamma = \frac{\delta L}{\delta\Gamma_\gamma{}^\gamma} = \frac{2}{n}\,\frac{\delta L}{\delta Q} = \frac{2-n}{2n}\,\sigma\,{}^*D\sigma\;. \tag{6.7}$$

In our formalism we may define a "new improved" energy–momentum current for scalar fields by

$$\Theta_\alpha := \Sigma_\alpha + e_\alpha\rfloor D\Delta\,. \tag{6.8}$$

After insertion of the field equation (6.4), we find for its trace the "weak" relation

$$\vartheta^\alpha \wedge \Theta_\alpha(\sigma) = \vartheta^\alpha \wedge \Sigma_\alpha(\sigma) + nD\Delta \cong nV(|\sigma|) - \frac{n-2}{2}\sigma\frac{\partial V(|\sigma|)}{\partial\sigma}. \tag{6.9}$$

Compared to (6.6), the kinetic term $L_\Box$ is absent in (6.9). Moreover, due to Euler's theorem for homogeneous functions, the $\sigma^{2n/(n-2)}$ piece in the potential drops out. For a polynomial potential $V(|\sigma|)$ of degree $p \leq 2n/(n-2)$, the operator dimensionality is then smaller than n (for $n \geq 4$). Therefore, the new trace is indeed "soft" in a momentum representation in the sense of Jackiw ([3], p. 213; cf. Kopczyński et al. [17]). Note that, for $n = 4$, a pure $\sigma^{2n/(n-2)}$ model is known to be renormalizable according to the criteria of power counting.

The necessary modification of our globally conserved current (5.10) is the following: We replace Σ_α by Θ_α and define the new 'improved' current by

$$\begin{aligned}
\tilde{\varepsilon}_{MA} &:= \xi^\alpha\Theta_\alpha + (e_\beta\rfloor\widehat{D\xi^\gamma})\,\not{\Delta}^{\gamma\beta}{}_\gamma + \frac{1}{2}(\xi\rfloor Q)\Delta \\
&= \xi^\alpha\Sigma_\alpha + \xi\rfloor D\Delta + (e_\beta\rfloor\widehat{D\xi^\gamma})\,\not{\Delta}^{\gamma\beta}{}_\gamma + \frac{1}{2}(\xi\rfloor Q)\Delta \\
&= \varepsilon_{\mathrm{C}} + \xi\rfloor D\Delta\,.
\end{aligned} \tag{6.10}$$

Provided the scaling relation

$$\ell_\xi(D\Delta) = \frac{n}{2}\omega\, D\Delta \tag{6.11}$$

holds, we finally obtain

$$d\tilde{\varepsilon}_{\,MA} = \frac{1}{2}\omega(\vartheta^\alpha \wedge \Theta_\alpha)\,. \tag{6.12}$$

This means that the divergence of the 'new improved' current (6.10) has indeed a soft trace also for scalar fields, as is required quantum–theoretically. If $\vartheta^\alpha \wedge \Theta_\alpha$ is vanishing, the scaling property (6.11) converts into one for the trace of the canonical energy-momentum current. In future we hope to find further physical motivations for this hypothetical scaling property.

7. **Emergence of the infinite-dimensional group of active diffeomorphisms**

In (3.5) we defined the hypermomentum current in metric-affine spacetime. In Minkowski spacetime we can expand the hypermomentum so as to exhibit the hypersurface $(n-1)$-form η_α. By using the improved energy-momentum tensor, we obtain an orbital representation:

$$\tilde{\Delta}^\alpha{}_\beta = \Delta^\alpha{}_\beta{}^\gamma \eta_\gamma \qquad , \qquad \Delta^\alpha{}_\beta{}^\gamma = x^\alpha \tilde{t}_\beta{}^\gamma\,. \tag{7.1}$$

The antisymmetric piece $\Delta_{[\alpha\beta]}{}^\gamma$ is the angular momentum tensor, whereas the shear components are given by the traceless part of the symmetric piece $\Delta_{(\alpha\beta)}{}^\gamma$. The integrated total 'charges' are given by (cf. the introduction)

$$D = \int x^\gamma\, \tilde{t}_\gamma{}^\alpha \eta_\alpha \qquad \text{(dilation charge)}\,, \tag{7.2}$$

$$K^\beta = \int [2x^\beta x^\gamma - g^{\beta\gamma}\, x^2]\, \tilde{t}_\gamma{}^\alpha \eta_\alpha \qquad \text{(proper conformal charges)}\,, \tag{7.3}$$

$$M_{[\beta\gamma]} = \int x_{[\beta} \tilde{t}_{\gamma]}{}^\alpha \eta_\alpha \qquad \text{(angular momentum charges)}\,, \tag{7.4}$$

$$\not{S}_{(\beta\gamma)} = \int [x_{(\beta} \tilde{t}_{\gamma)}{}^\alpha - \frac{1}{n} g_{\beta\gamma}\, x^\delta \tilde{t}_\delta{}^\alpha]\eta_\alpha \qquad \text{(shear charges)}\,. \tag{7.5}$$

Using Bohr's principle of correspondence, Ogievetsky [18] has shown that the quantized system of shear $\not{S}_{(\beta\gamma)}$ and proper conformal K^β charges does not close algebraically and generates the infinite algebra of diffeomorphism charges in n dimensions:

$${}^n L_\alpha^{<p_0,\ldots,p_{n-1}>\gamma\delta\epsilon\xi\eta\theta\cdots} = \int \prod_{i=0}^{n-1} (x_i)^{p_i}\; \tilde{t}_\alpha{}^\beta \eta_\beta\,. \tag{7.6}$$

The algebraic relations are preserved anholonomically. Thus, a metric-affine spacetime which admits a conformal symmetry will have its frames locally invariant (in the *active* operational sense of [19],[20]) under the group of analytical diffeomorphisms. This result overlaps with the fact that we have included in our affine gauge approach local translations, i.e. active diffeomorphisms, except that whereas the latter are only infinitesimal (their generators do not form a Lie algebra anyhow), the Ogievetsky transformations can be integrated to finite diffeomorphisms, provided we restrict to constant parameters and thus do not require an infinite set of commutators. The emergence of an explicit infinite Lie algebra may make it possible to treat conformal fields in n-dimensions similarly to what is done in the special case of two dimensions. In $n = 2$, there is an infinite–dimensional conformal algebra which is isomorphic to the algebra of analytic two–dimensional diffeomorphisms [21]. In two dimensions this feature constrains the fields and leads to the highly restrictive 'fusion' rules [21], which have recently put 2-dimensional conformal field theory into the focus of interest of statistical mechanics and string theory.

The Ogievetsky algebra in n dimensions is conceptually the analog of the de Witt algebra in two dimensions and should possess a quantum extension with central charges as in the Virasoro algebra. Neither this extension nor the representation theory have been investigated to date.

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References

[1] R. Penrose and W. Rindler: *Spinors in spacetime*, Vol. 2. (Cambridge University Press 1984); see also L.B. Szabados: Canonical pseudotensors, Sparling's form and Noether currents, preprint KFKI-1991-29/B, Hungarian Acad. Sci. Budapest (1991).

[2] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne'eman, *Found. Phys.* **19**, 1075 (1989); *Phys. Rep.* (to be published); J.D. McCrea, *Class. Quantum Grav.* **9**, 553 (1992).

[3] R. Jackiw, in: *Lectures on Current Algebra and its Applications*, B. Treiman, R. Jackiw, and D.J. Gross, eds. (Princeton University Press, Princeton 1972) p. 97, see in particular p. 212.

[4] R. P. Wallner: "Feldtheorie im Formenkalkül", Ph.D.-thesis, University of Vienna 1982.

[5] Y. Ne'eman, in: *Spinors in physics and geometry*, Trieste, 11-13 Sept. 1986, A. Trautman and G. Furlan eds. (World Scientific, Singapore 1987), p. 313; Y. Ne'eman, *Ann. Inst. Henri Poincaré* **A28**, 369 (1978).

[6] J. M. Nester, in: *An Introduction to Kaluza-Klein Theories*, Workshop Chalk River/Deep River, Ontario 11-16 Aug. 1983, H. C. Lee, eds. (World Scientific, Singapore 1984), p.83.

[7] W. Kopczyński, *Ann. Phys.* (N.Y.), **203**, 308 (1990).

[8] F.W. Hehl and E.W. Mielke, *Wiss. Zeitschr. d. Friedrich-Schiller-Universität Jena* **39**, 58 (1990).

[9] I.M. Benn and W.P. Wood, *J. Math. Phys.* **33**, 2765 (1992).

[10] S. Coleman, *Phys. Rev.* **177**, 2239, 2247 (1969).

[11] A. Trautman, *Bull. Acad. Polon. Sci., Sér. sciences math., astr. et phys.* **20**, 895 (1972).

[12] K.P. Tod, in: *Twistors in Mathematics and Physics*, T. N. Bailey and R. J. Baston eds. (Cambridge University Press, Cambridge 1990), p. 164.

[13] J. Audretsch, F.W. Hehl, and C. Lämmerzahl in: *Proceedings of the Bad Honnef School on Gravitation*, J. Ehlers and G. Schäfer, eds. Lecture Notes in Physics, (Springer-Verlag, Berlin, to be published 1992).

[14] R. Penrose, *Proc. Roy. Soc.* (London) **A381**, 53 (1982); R. Penrose, in: *Global Riemannian Geometry*, T.J. Willmore and N.J. Hitchin, eds. (Ellis Harwood Limited, New York 1984), p. 203.

[15] E. W. Mielke, *Gen. Rel. Grav.* **8**, 321 (1977).

[16] C.J. Isham, Abdus Salam, and J. Strathdee, *Phys. Rev.* **D2**, 685 (1970).

[17] W. Kopczyński, J.D. McCrea, and F.W. Hehl, *Phys. Lett.* **A128**, 313 (1988).

[18] V. I. Ogievetsky, *Lett. Nuovo Cimento* **8**, 988 (1973).

[19] F. Hehl, P. v. Heyde, G. D. Kerlick and J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).

[20] Y. Ne'eman, in: *Differential Geometrical Methods in Mathematical Physics*, K. Bleuler, H. R. Petry, and A. Reetz, eds. Lecture Notes in Mathematics, Vol. 676 (Springer-Verlag, Berlin 1978), p.189.

[21] M. Kaku, *Strings, Conformal Fields, and Topology*, (Springer-Verlag, New-York 1991).